(3) Methods for solving integer programs:
Cutting planes & Branch-and-bound

Exercise 3.1. Using the ℓ-method, find the optimal solution of the linear program:

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 \\
\text{subject to} \quad & -x_1 + x_2 \leq 1, \\
& 3x_1 - 4x_2 \leq 6, \\
& x_1 + x_2 \leq 4, \\
& x_1 + 3x_2 \leq 9, \\
& x_2 \leq \tfrac{5}{2}, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Exercise 3.2. Find the optimal solution of the following integer linear programs using Gomory's first cutting plane method:

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & 3x_1 + 6x_2 \leq 10, \\
& x_1, x_2 \geq 0, \\
& x_1, x_2 \in \mathbb{Z},
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 7, \\
& -3x_1 + x_2 \leq -1, \\
& x_1, x_2 \geq 0, \\
& x_1, x_2 \in \mathbb{Z},
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & x_1 - x_2 \\
\text{s.t.} \quad & -\tfrac{1}{3}x_1 + x_2 \leq \tfrac{1}{3}, \\
& x_1 - \tfrac{1}{3}x_2 \leq \tfrac{1}{3}, \\
& x_1, x_2 \geq 0, \\
& x_1, x_2 \in \mathbb{Z}.
\end{aligned}
$$

Exercise 3.3. Solve the last integer linear program from Exercise 3.2 using Gomory's second cutting plane method.
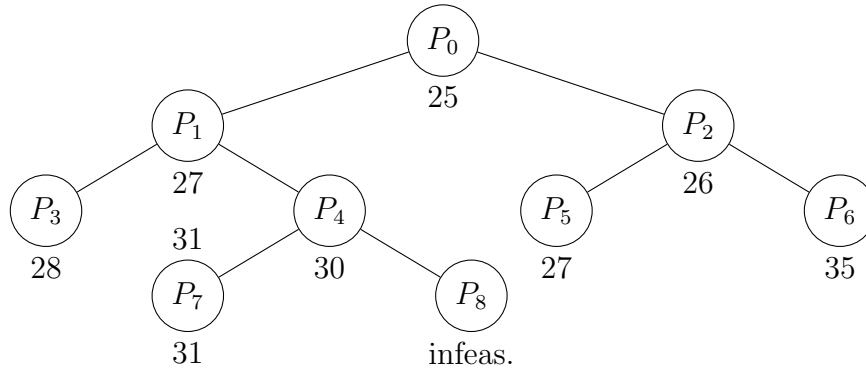
Exercise 3.4.

(a) How can we detect unbounded linear programs when using the ℓ-method?

(b) How can we detect an infeasible linear relaxation, or an infeasible integer program when using Gomory's cutting plane methods?

Exercise 3.5. Find a Chvátal-Gomory cut for the set

$$
M = \{x \in \mathbb{Z}^5 : 9x_1 + 12x_2 + 8x_3 + 17x_4 + 13x_5 \geq 50, x \geq 0\}
$$

cutting off the point $(0, \tfrac{25}{6}, 0, 0, 0)$.

Exercise 3.6. Consider a minimization integer program with the following tree appearing in the branch-and-bound computation (numbers below the nodes represent the optimal value of the corresponding linear relaxation, numbers above the nodes the optimal value corresponding to an integer solution):

Give the tightest possible bounds on the optimal value. Which nodes can be pruned and which must be explored further?

Exercise 3.7. Solve the following integer program using the branch-and-bound method:

(a) the last program from Exercise 3.2,

(b) integer linear program:
$$\begin{aligned}
\max \quad & 13x_1 + 8x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 10, \\
& 5x_1 + 2x_2 \leq 20, \\
& x_1, x_2 \in \mathbb{N}_0.
\end{aligned}$$

Exercise 3.8. Use various preprocessing techniques (tightening the bounds, checking redundancy or infeasibility, fixing the variables) to simplify the integer linear program:

$$\begin{aligned}
\max \quad & 2x_1 + x_2 - x_3 \\
\text{subject to} \quad & 5x_1 - 2x_2 + 8x_3 \leq 15, \\
& 8x_1 + 3x_2 - x_3 \geq 9, \\
& x_1 + x_2 + x_3 \leq 6, \\
& x_1 \in [0, 3], \ x_2 \in [0, 1], \ x_3 \geq 1, \\
& x_1, x_2, x_3 \in \mathbb{Z}.
\end{aligned}$$