

(3) Methods for solving integer programs: Cutting planes & Branch-and-bound

Exercise 3.1. Using the ℓ -method, find the optimal solution of the linear program:

$$\begin{array}{ll}\max & x_1 + 2x_2 \\ \text{subject to} & -x_1 + x_2 \leq 1, \\ & 3x_1 - 4x_2 \leq 6, \\ & x_1 + x_2 \leq 4, \\ & x_1 + 3x_2 \leq 9, \\ & x_2 \leq \frac{5}{2}, \\ & x_1, x_2 \geq 0.\end{array}$$

Exercise 3.2. Find the optimal solution of the following integer linear programs using Gomory's first cutting plane method:

$$\begin{array}{ll}\max & x_1 + x_2 \\ \text{s. t.} & 3x_1 + 6x_2 \leq 10, \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{Z},\end{array}$$

$$\begin{array}{ll}\max & x_2 \\ \text{s. t.} & 2x_1 + x_2 \leq 7, \\ & -3x_1 + x_2 \leq -1, \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{Z},\end{array}$$

$$\begin{array}{ll}\max & x_1 - x_2 \\ \text{s. t.} & -\frac{1}{3}x_1 + x_2 \leq \frac{1}{3}, \\ & x_1 - \frac{1}{3}x_2 \leq \frac{1}{3}, \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{Z}.\end{array}$$

Exercise 3.3. Solve the last integer linear program from Exercise 3.2 using Gomory's second cutting plane method.

Exercise 3.4.

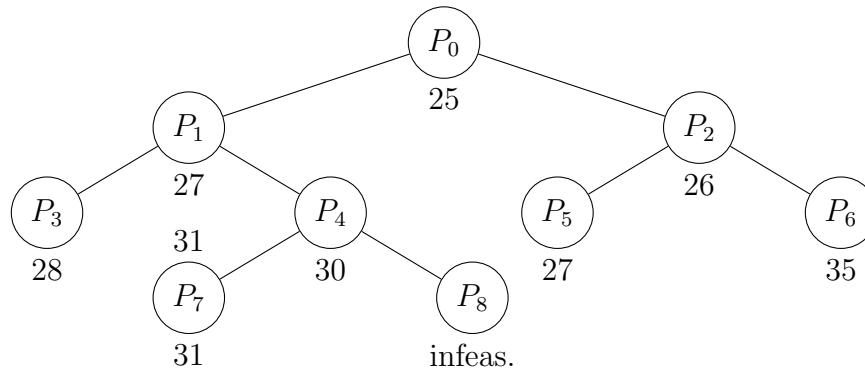
- (a) How can we detect unbounded linear programs when using the ℓ -method?
- (b) How can we detect an infeasible linear relaxation, or an infeasible integer program when using Gomory's cutting plane methods?

Exercise 3.5. Find a Chvátal-Gomory cut for the set

$$M = \{x \in \mathbb{Z}^5 : 9x_1 + 12x_2 + 8x_3 + 17x_4 + 13x_5 \geq 50, x \geq 0\}$$

cutting off the point $(0, \frac{25}{6}, 0, 0, 0)$.

Exercise 3.6. Consider a minimization integer program with the following tree appearing in the branch-and-bound computation (numbers below the nodes represent the optimal value of the corresponding linear relaxation, numbers above the nodes the optimal value corresponding to an integer solution):



Give the tightest possible bounds on the optimal value. Which nodes can be pruned and which must be explored further?

Exercise 3.7. Solve the following integer program using the branch-and-bound method:

- (a) the last program from Exercise 3.2,
- (b) integer linear program:
$$\begin{array}{ll} \max & 13x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10, \\ & 5x_1 + 2x_2 \leq 20, \\ & x_1, x_2 \in \mathbb{N}_0. \end{array}$$

Exercise 3.8. Use various preprocessing techniques (tightening the bounds, checking redundancy or infeasibility, fixing the variables) to simplify the integer linear program:

$$\begin{array}{ll} \max & 2x_1 + x_2 - x_3 \\ \text{subject to} & 5x_1 - 2x_2 + 8x_3 \leq 15, \\ & 8x_1 + 3x_2 - x_3 \geq 9, \\ & x_1 + x_2 + x_3 \leq 6, \\ & x_1 \in [0, 3], x_2 \in [0, 1], x_3 \geq 1, \\ & x_1, x_2, x_3 \in \mathbb{Z}. \end{array}$$